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Applied Mathematics Letters 18 (2005) 563–569

Applied Mathematics Letters

www.elsevier.com/locate/aml

Abelian integrals and period functions for quasihomogeneous Hamiltonian vector fields☆

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Received 17 March 2004; accepted 12 August 2004

Abstract

In this paper, we study the number of zeros of Abelian integrals and the monotonicity of period functions for planar quasihomogeneous Hamiltonian vector fields. The result for Abelian integrals extends the recent work of Li et al. [C. Li, W. Li, J. Llibre, Z. Zhang, Polynomial systems: A lower bound for the weakened 16th Hilbert problem, Extracta Math. 16 (3) (2001) 441–447] and Llibre and Zhang [J. Llibre, X. Zhang, On the number of limit cycles for some perturbed Hamiltonian polynomial systems, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 8 (2) (2001) 161–181].

Keywords: Quasihomogeneous vector field; Abelian integral; Period function

1. Introduction

Let $f(x, y), g(x, y)$ and $H(x, y)$ be real polynomials of two variables x, y from the polynomial ring $\mathbb{R}[x, y]$ and $\max\{\deg f(x, y), \deg g(x, y)\} = s$. A closed connected component of a level set $H(x, y) = h$ is denoted by Γ_h. Let

$$\omega = -f(x, y)\mathrm{d}y + g(x, y)\mathrm{d}x, \qquad I(h) = \oint_{\Gamma_h} \omega. \tag{1.1}$$

☆ Supported by NSF of China and Quangdong Natural Science Foundation (No. 04009794).
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0893-9659/$ - see front matter
doi:10.1016/j.aml.2004.08.008

The problem of finding an upper bound for the number of isolated zeros of the *Abelian integral* $I(h)$ is called *the weakened Hilbert 16th problem* [3]. Consider the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x} \tag{1.2}$$

and the corresponding perturbed system

$$\dot{x} = \frac{\partial H}{\partial y} + \epsilon f(x, y), \qquad \dot{y} = -\frac{\partial H}{\partial x} + \epsilon g(x, y), \tag{1.3}$$

where we assume that system (1.2) has at least one center. It is well known that, if $I(h) \not\equiv 0$, then the total number of isolated zeros of $I(h)$ (taking into account their multiplicities) is an upper bound for the number of limit cycles in (1.3) which emerge from the periodic orbits of the unperturbed Hamiltonian system (1.2) as $\epsilon \to 0$.

Many authors have contributed to the investigations of this problem; see [4,5] and references therein. Recently, Li et al. [1], Llibre and Zhang [2] have studied the Hamiltonians

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{m}x^{2m}, \tag{1.4}$$

$$H(x, y) = \frac{1}{2n}x^{2n} + \frac{1}{2m}y^{2m}, \tag{1.5}$$

respectively; they obtained the maximum number of zeros of the corresponding Abelian integrals, which is also the upper bounds for the number of limit cycles of (1.3) which emerge from the periodic orbits of system (1.2) as $\epsilon \to 0$.

Let $T(h)$ be the period of the periodic orbit Γ_h of system (1.2). It follows from the first equation of system (1.2) that

$$T(h) = \oint_{\Gamma_h} \frac{1}{H_y} \mathrm{d}x. \tag{1.6}$$

A center is called *isochronous* if $T(h)$ is identically equal to a constant. There has been a substantial amount of work devoted to understanding the behavior of $T(h)$; see for instance [6,7] and references therein.

This paper is concerned with the number of zeros of Abelian integrals and the monotonicity of period functions for planar quasihomogeneous Hamiltonian vector fields. We first recall the following definition.

Definition 1.1. A polynomial $H(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be quasihomogeneous of weighted degree l with index (α, β) if, for any $\rho > 0$, we have $H(\rho^\alpha x, \rho^\beta y) = \rho^l H(x, y)$. The indices α, β are called the weights of the variable x and y respectively.

The system (1.2) is called a quasihomogeneous Hamiltonian vector field if $H(x, y)$ is quasihomogeneous.

Example 1.2. The Hamiltonian (1.4) (resp. (1.5)) is a quasihomogeneous of weighted degree $2m$ (resp. $2mn$) with the weights $\alpha = 1, \beta = m$ (resp. $\alpha = m, \beta = n$).

In this paper, we always assume $\alpha, \beta, l \in \mathbb{Q}^+$. Let $H = \sum \eta_{k_1 k_2} x^{k_1} y^{k_2}$, then the condition of quasihomogeneity means that all the indices of non-null lie on a plane $\{(k_1, k_2) \mid k_1\alpha + k_2\beta = l\}$. Without loss of generality, we suppose that $\alpha, \beta, l \in \mathbb{Z}^+$ and $((\alpha, \beta), l) = 1$, where (α, β) denotes the maximal common factor of α and β.

If $\alpha = \beta$, then $k_1 + k_2 = l/\alpha$. By the above assumption, we have $\alpha = \beta = 1$. In this case, $H(x, y)$ is called *homogeneous*.

The *Euler field* is the derivation $X = \alpha x \frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y}$. By construction,

$$XH = \alpha x \frac{\partial H}{\partial x} + \beta y \frac{\partial H}{\partial y} = lH(x, y) \tag{1.7}$$

for any quasihomogeneous polynomial $H(x, y)$ (the Euler identity).

The main results of this paper are split into two parts. In Section 2, the equations satisfied by Abelian integrals are derived for the quasihomogeneous Hamiltonian vector field (1.2). We obtain the maximum number of isolated zeros (taking into account their multiplicities) of the corresponding Abelian integral $I(h)$. This generalizes the results in [1,2]. In Section 3, we study the monotonicity of period functions of periodic orbits for the quasihomogeneous Hamiltonian system (1.2). It is proved that the period function is monotonic if $H(x, y)$ is not a quadratic polynomials.

Conventions. From now on, we always suppose that $H(x, y)$ is a quasihomogeneous Hamiltonian in x, y, $(0, 0)$ is a center of system (1.2) and Γ_h is defined in $(0, +\infty)$.

2. The number of zeros of Abelian integrals

We first recall the following preliminary results which will be used later on.

Gelfand–Leray Formula. *If a pair of polynomial 1-form ω, θ satisfies the identity* $\mathrm{d}\omega = \mathrm{d}H \wedge \theta$, *then for any continuous family of cycle $\Gamma_h \subset \{(x, y) \mid H(x, y) = h\}$,*

$$\frac{\mathrm{d}}{\mathrm{d}h} \oint_{\Gamma_h} \omega = \oint_{\Gamma_h} \theta. \tag{2.1}$$

Descartes' Theorem (*See for Instance [2]*). *Consider the real polynomial function $p(x) = a_{i_1} x^{i_1} + a_{i_2} x^{i_2} + \cdots + a_{i_n} x^{i_n}$ with $0 \le i_1 < i_2 < \cdots < i_n$ and $a_{i_j} \ne 0$ for $j \in \{1, 2, \ldots, n\}$. When $a_{i_j} a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ form the changes of sign. If the number of variations of sign is r, then $p(x)$ has at most r positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such way that $p(x)$ has exact $n - 1$ positive roots.*

The main results of this section is the following.

Theorem 1. *Suppose that $H(x, y)$ is a quasihomogeneous polynomial of weighted degree l with the weights α, β, then the number of zeros (taking into account their multiplicities) of Abelian integral $I(h)$, defined in* (1.1), *does not exceed* (i) $s - 1$ *for homogeneous cases* $\alpha = \beta = 1$, (ii) $s(s + 3)/2 - 1$ *for* $\alpha \ne \beta$, $s < \mu$, (iii) $\mu(2s - \mu + 3)/2 - 2$ *for* $\alpha \ne \beta$, $s \ge \mu$, *where* $(\alpha, \beta) = \nu, \alpha = p\nu, \beta = q\nu$, $\mu = \max\{p, q\}$.

Proof. Set

$$\omega_{ij} = x^i y^j \mathrm{d}x, \qquad I_{ij} = I_{ij}(h) = \oint_{\Gamma_h} \omega_{ij} \mathrm{d}x.$$

Using the Euler identity (1.7) and $\mathrm{d}H = H_x \mathrm{d}x + H_y \mathrm{d}y$, we have

$$
\begin{aligned}
lH\mathrm{d}\omega_{ij} &= jlHx^iy^{j-1}\mathrm{d}y\wedge\mathrm{d}x = j(\alpha xH_x+\beta yH_y)x^iy^{j-1}\mathrm{d}y\wedge\mathrm{d}x\\
&= -j\alpha x^{i+1}y^{j-1}(\mathrm{d}H-H_y\mathrm{d}y)\wedge\mathrm{d}y + j\beta x^iy^j(\mathrm{d}H-H_x\mathrm{d}x)\wedge\mathrm{d}x\\
&= -j\alpha x^{i+1}y^{j-1}\mathrm{d}H\wedge\mathrm{d}y + j\beta x^iy^j\mathrm{d}H\wedge\mathrm{d}x\\
&= \mathrm{d}H\wedge(-\alpha x^{i+1}\mathrm{d}y^j) + j\beta x^iy^j\mathrm{d}H\wedge\mathrm{d}x\\
&= \mathrm{d}H\wedge\left(d(-\alpha x^{i+1}y^j)+\alpha(i+1)x^iy^j\mathrm{d}x\right) + j\beta x^iy^j\mathrm{d}H\wedge\mathrm{d}x\\
&= \mathrm{d}H\wedge\left(d(-\alpha x^{i+1}y^j)+(\alpha(i+1)+\beta j)\,x^iy^j\mathrm{d}x\right).
\end{aligned}
$$

Integrating over closed ovals Γ_h (so that the exact form $\mathrm{d}(-\alpha x^{i+1}y^j)$ disappears) and using the Gelfand–Leray formula (2.1), we conclude

$$lhI'_{ij} = (\alpha(i+1)+\beta j)\,I_{ij}. \tag{2.2}$$

Here the prime $'$ denotes the derivative with respect to h. The Eq. (2.2) implies

$$I_{ij}(h) = c_{ij}h^{\frac{\alpha(i+1)+\beta j}{l}}, \tag{2.3}$$

where c_{ij} is a real constant. Since

$$\oint_{\Gamma_h} y^j\mathrm{d}y = 0, \qquad \oint_{\Gamma_h} x^iy^j\mathrm{d}y = -\frac{i}{j+1}I_{i-1,j+1}, \quad i\ge 1,$$

we obtain that $I(h)$ can be expressed as

$$I(h) = \sum_{i+j=1}^{s}\delta_{ij}I_{ij} = \sum_{i+j=1}^{s}\widetilde{\delta}_{ij}h^{\frac{\alpha(i+1)+\beta j}{l}} = h^{\frac{\alpha}{l}}\sum_{i+j=1}^{s}\widetilde{\delta}_{ij}(\widetilde{h})^{\gamma_{ij}},$$

where we use the identity $I_{00}\equiv 0$ and $\gamma_{ij} = \alpha i+\beta j$, $\widetilde{h} = h^{1/l}$, $\widetilde{\delta}_{ij} = c_{ij}\delta_{ij}$. To obtain the number of zeros of $I(h)$ by Descartes' Theorem, we should estimate how many different values of γ_{ij}'s can be taken for $1\le i+j\le s$. It will be done by using the same arguments as the proof of Theorem 2 in [2].

Let $D_{\alpha\beta s}$ be the number of γ_{ij}'s with different values for $1\le i+j\le s$. Then the number of zeros of Abelian integrals $I(h)$ does not exceed $D_{\alpha\beta s}-1$ by Descartes' Theorem.

For homogeneous cases $\alpha=\beta=1$, we have $\gamma_{ij}=i+j$, which implies $D_{\alpha\beta s}\le s$.

Assume $\alpha\ne\beta$. Without loss of generality, we suppose $\alpha<\beta$. The rest of the proof is split into two cases.

Case 1. $(\alpha,\beta)=1$, $\alpha<\beta$.

Subcase (1). $s<\beta$. We will prove that, *if either* $i_1\ne i_2$ *or* $j_1\ne j_2$, *then* $\gamma_{i_1j_1}\ne\gamma_{i_2,j_2}$, which implies that

$$D_{\alpha\beta s} = 2+3+\cdots+(s+1) = \frac{s(s+3)}{2}.$$

To prove the above results, assume the contrary, i.e., $\gamma_{i_1j_1}=\gamma_{i_2j_2}$, which means

$$(i_1-i_2)\alpha+(j_1-j_2)\beta = 0. \tag{2.4}$$

It follows from (2.4) that $i_1\ne i_2$ if and only if $j_1\ne j_2$.

Suppose $i_1\ne i_2$, then we get from $(\alpha,\beta)=1$ and (2.4) that β divides i_1-i_2. This is impossible because $0\le i_1,i_2\le s<\beta$ and $i_1-i_2\ne 0$. Therefore, $\gamma_{i_1j_1}\ne\gamma_{i_2j_2}$ if $i_1\ne i_2$.

Subcase (2). $s\ge\beta>\alpha$.

If $i \geq \beta$, then i can be expressed as $i = t\beta + i^*$, where $t \in \mathbb{Z}^+, i^* \in \mathbb{Z}^+ \cup \{0\}, 0 \leq i^* \leq \beta - 1$. Let $j^* = t\alpha + j$. Since $\alpha < \beta$, we have $i + j = t\beta + i^* + j > t\alpha + i^* + j = i^* + j^*$. On the other hand, $\gamma_{ij} = \alpha i + \beta j = \alpha(t\beta + i^*) + \beta j = \alpha i^* + \beta j^* = \gamma_{i^* j^*}$. This means, for arbitrary $i, j, \beta < i + j \leq s, i \geq \beta$, there exists $i^*, j^*, 0 \leq i^* < \beta$, such that $\gamma_{ij} = \gamma_{i^* j^*}$.

If $i < \beta$ and $\beta \leq i + j \leq s$, then by the same arguments as Subcase (1), we have $\gamma_{ij} \neq \gamma_{i_1 j_1}$ for any $i_1 < \beta$ and $i_1 \neq i$ or $j_1 \neq j$.

By the above discussions, we obtain that, if $i + j = r$ and $\beta \leq r \leq s$, then γ_{ij} with different values, which is also different from any $\gamma_{i_1 j_1}$ for $i_1 < \beta$ and $i_1 + j_1 \neq r$, are $\gamma_{0r}, \gamma_{1,r-1}, \ldots, \gamma_{\beta-1,r-\beta+1}$. Therefore, the number of γ_{ij} with different values, which is also different from $\gamma_{i'j'}$, is $\beta(s - \beta + 1)$, where $\beta \leq i + j \leq s, i' + j' \leq \beta$.

Using the same arguments as Subcase (1), we know that $\gamma_{ij} \neq \gamma_{i_1 j_1}$ for $i + j \leq \beta, i_1 + j_1 \leq \beta$. Hence,

$$D_{\alpha\beta s} = 2 + 3 + \cdots + \beta + \beta(s - \beta + 1) = \frac{\beta(2s - \beta + 3)}{2} - 1.$$

Case 2. $(\alpha, \beta) = \nu > 1$.

In this case, $\gamma_{ij} = \nu(pi + qj)$. Using the same arguments as Case 1, we get the results.

Theorem 2. *If $H(x, y) = H(-x, y)$ and $H(x, y) = H(x, -y)$ hold for a quasihomogeneous system, then*

(1) *The number of zeros of Abelian integrals $I(h)$, defined in* (1.1)*, does not exceed $B(s)$, where $B(s)$ is equal to* (i) λ *for homogeneous cases* $\alpha = \beta = 1$, (ii) $\lambda(\lambda + 3)/2$ *for* $\alpha \neq \beta, \lambda < \mu$, (iii) $\mu(2\lambda - \mu + 3)/2 - 1$ *for* $\alpha \neq \beta, \lambda \geq \mu$. *Here μ, ν, are defined as in Theorem 1, $\lambda = [(s - 1)/2]$ and $[r]$ denotes the entire part of r.*

(2) *There exists* $f(x, y), g(x, y) \in \mathbb{R}[x, y], \max\{\deg f(x, y), \deg g(x, y)\} = s$, *such that $I(h)$ has exact $B(s)$ zeros in $(0, +\infty)$, i.e., the corresponding system* (1.3) *has exact $B(s)$ limit cycles which emerge from the periodic orbits of system* (1.2) *as $\epsilon \to 0$.*

This theorem agrees with the results in [1,2] for the Hamiltonian (1.4) and (1.5).

Proof. If i and j are not even simultaneously, then $I_{i,j+1} \equiv 0$ by symmetry. Therefore,

$$\begin{aligned} I(h) &= \sum_{i+j=1}^{s} \delta_{ij} I_{i,j} = \sum_{i+j=0}^{s-1} \delta_{i,j+1} I_{i,j+1} = \sum_{2i+2j=0}^{2\lambda} \delta_{2i,2j+1} I_{2i,2j+1} \\ &= \sum_{i+j=0}^{\lambda} \widetilde{\delta}_{2i,2j+1} h^{\frac{\alpha(2i+1)+2j\beta}{l}} = h^{\alpha/l} \sum_{i+j=0}^{\lambda} \widetilde{\delta}_{2i,2j+1} (h^{2/l})^{\alpha i + \beta j}. \end{aligned}$$

Using the same arguments as in the proof of Theorem 1, we get (1).

On the other hand, it follows from Green's formula that

$$I_{2i,2j+1} = \int\int_{int\Gamma_h} x^{2i} y^{2j} \mathrm{d}x\mathrm{d}y \neq 0, \tag{2.5}$$

which yields $c_{2i,2j+1} \neq 0$ in (2.3). So there exists $f(x, y), g(x, y)$ (hence $\widetilde{\delta}_{2i,2j+1}$), such that $I(h)$ has $B(s)$ positive zeros in $(0, +\infty)$ by using Descartes' Theorem. □

3. The monotonicity of period functions

In this section, we study the monotonicity of period functions of periodic orbits of quasihomogeneous vector fields (1.2). Since $H_y \frac{\partial y}{\partial h} = 1$, we have

$$I'_{01}(h) = \oint_{\Gamma_h} \frac{\partial y}{\partial h} \mathrm{d}x = \oint_{\Gamma_h} \frac{1}{H_y} \mathrm{d}x.$$

We get from (1.6) and (2.3) that

$$T(h) = I'_{01}(h) = \frac{c_{01}(\alpha+\beta)}{l} h^{(\alpha+\beta-l)/l}, \qquad h \in (0, +\infty). \tag{3.1}$$

It follows from (2.5) that $c_{01} \neq 0$ in (3.1). The main result of this section is the following.

Theorem 3. (i) *The center* $(0,0)$ *of quasihomogeneous vector fields* (1.2) *is isochronous if and only if* $H(x,y)$ *is a quadratic homogeneous polynomial, i.e.,* $l = 2, \alpha = \beta = 1$.

(ii) *For any case of others, the period function* $T(h)$ *is monotonic.*

Proof. (i) It follows from (3.1) that the center (0, 0) is isochronous if and only if $\alpha + \beta = l$.

(a) Suppose that (0, 0) is isochronous and $H(x,y) = \sum \eta_{k_1k_2} x^{k_1} y^{k_2}$. The condition of quasihomogeneity means that the index (k_1, k_2) lies on the plane

$$k_1\alpha + k_2\beta = \alpha + \beta = l. \tag{3.2}$$

If $H(x,y)$ is homogenous, then without loss of generality we assume $\alpha = \beta = 1$. By (3.2), we get $k_1 + k_2 = 2$, which implies $(k_1,k_2) = (2,0), (1,1), (0,2)$. This yields $H(x,y) = \eta_{20}x^2 + \eta_{11}xy + \eta_{02}y^2$. The corresponding Hamiltonian system with center can be reduced into the linear system $\dot{x} = y$, $\dot{y} = -x$, which has a isochronous center at (0, 0).

If $\alpha \neq \beta$, then without loss generality we suppose $\alpha < \beta$. Using (3.2) again, we have

$$k_2 = 1 + \frac{\alpha}{\beta}(1 - k_1) = \begin{cases} k_2 \notin \mathbb{Z}^+ \cup \{0\}, & \text{if } k_1 = 0, \\ k_2 = 1, & \text{if } k_1 = 1, \\ k_2 = 0, & \text{if } k_1 \geq 2, \beta + \alpha(1-k_1) = 0, \\ k_2 \notin \mathbb{Z}^+ \cup \{0\} & \text{if } k_1 \geq 2, \beta + \alpha(1-k_1) \neq 0, \end{cases}$$

which shows that $H(x,y)$ has the form $H(x,y) = \eta_{11}xy + \eta_{1+\beta/\alpha} x^{1+\beta/\alpha}$. The corresponding Hamiltonian system is

$$\dot{x} = \eta_{11}x, \qquad \dot{y} = -\eta_{11}y - \left(1 + \frac{\beta}{\alpha}\right) \eta_{1+\beta/\alpha} y^{\beta/\alpha}.$$

The critical point (0, 0) is not a center of the above system.

(b) Conversely, if $l = 2, \alpha = \beta = 1$, then the corresponding Hamiltonian system should be reduced to the form $\dot{x} = y, \dot{y} = -x$, which has a isochronous center at (0, 0).

(ii) By (3.1),we conclude that $T'(h) \neq 0$ in $h \in (0, +\infty)$ if $\alpha + \beta \neq l$. The result (ii) follows. □

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